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THE TVERBERG–VREĆICA PROBLEM AND THE COMBINATORIAL GEOMETRY ON VECTOR BUNDLES

BY

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ABSTRACT

It is shown that many classical and many new combinatorial geometric results about finite sets of points in R^d, specially the theorems of Tverberg type, can be generalized to the case of vector bundles, where they become combinatorial geometric statements about finite families of continuous cross-sections. The well known Tverberg–Vrećica conjecture is interpreted as a result of this type and its partial solution is obtained with the aid of the parametrized, ideal-valued, cohomological index theory. In the same spirit, classical "nonembeddability" and "coincidence" results like $K_{3,3} \not\hookrightarrow R^2$ have higher dimensional analogues. A new ingredient is that the coincidence condition is often interpreted as the existence of a common affine k-dimensional transversal, which reduces to the classical case for $k = 0$.

1. Introduction

One of the objectives of this paper is to demonstrate that the euclidean space R^d, the usual ambient space for objects studied by combinatorial geometry, can be replaced by a vector bundle $R^d \to E \to B$ which is viewed as a family of d-dimensional vector spaces parametrized by a base space B. This is demonstrated by showing that there exist statements, the Tverberg–Vrećica conjecture being

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the main example, which are naturally rephrased and often solved in this broader context.

The Tverberg or Tverberg–Vrećica problem is one of the central conjectures in combinatorial geometry. It is a natural "supremum" of a large class of Tverberg type statements and results about combinatorial partitions of masses in R^d. At the origin of this conjecture, although it may not be obvious at the first sight, are Helly's convexity theorem and its relative Radon's theorem (Figure 1).

CONJECTURE 1.1 (Tverberg–Vrećica conjecture, [26]): *Assume that* $0 \leq k \leq d-1$ *and let* $S_0, S_1, \ldots, S_k$ *be a collection of finite sets in* R^d *of given cardinalities,* $|S_i| = (r_i - 1)(d - k + 1) + 1, i = 0, 1, \ldots, k$. *Then* S_i *can be partitioned into* r_i *nonempty sets,* $S_i^1, \ldots, S_i^{r_i}$, *so that for some k-dimensional affine subspace* $P \subset R^d$

$$P \cap \operatorname{conv}(S_i^j) \neq \emptyset \quad \textit{for each pair } (i,j), \quad 0 \leq i \leq k, \quad 0 \leq j \leq r_i.$$

The main result of this paper is a partial positive answer to this conjecture.

THEOREM 1.2: *The Tverberg–Vrećica conjecture is true under the condition that both d and k are odd integers and* $r_i = q$ *for all* $i = 0, \ldots, k$, *where q is an odd prime number.*

The proof of this theorem is postponed for section 4. There we formulate a more general, nonlinear version of Conjecture 1.1 and establish the corresponding more general version of Theorem 1.2.

Helge Tverberg formulated Conjecture 1.1 at the 1989 Symposium on Combinatorics and Geometry in Stockholm. It appeared in print in [26] where Tverberg and Vrećica established a slightly weakened version of the conjecture in the case $k = d - 2$. Tverberg was apparently motivated by the observation that both the well known Tverberg theorem and the "central transversal theorem", Theorems 1.3 and 1.4 below, are consequences of Conjecture 1.1. Figure 1 shows the relationship among these results and the other well known combinatorial geometric statements.

THEOREM 1.3 (H. Tverberg, [25]): *Every set* $K = \{a_j\}_{j=0}^{(q-1)(d+1)} \subset R^d$, *consisting of* $(q-1)(d+1)+1$ *elements, can be partitioned into q nonempty, disjoint pieces* $K_1, \ldots, K_q$, *so that the corresponding convex hulls have a nonempty intersection*

$$\bigcap_{i=1}^{q} \operatorname{conv}(K_i) \neq \emptyset.$$

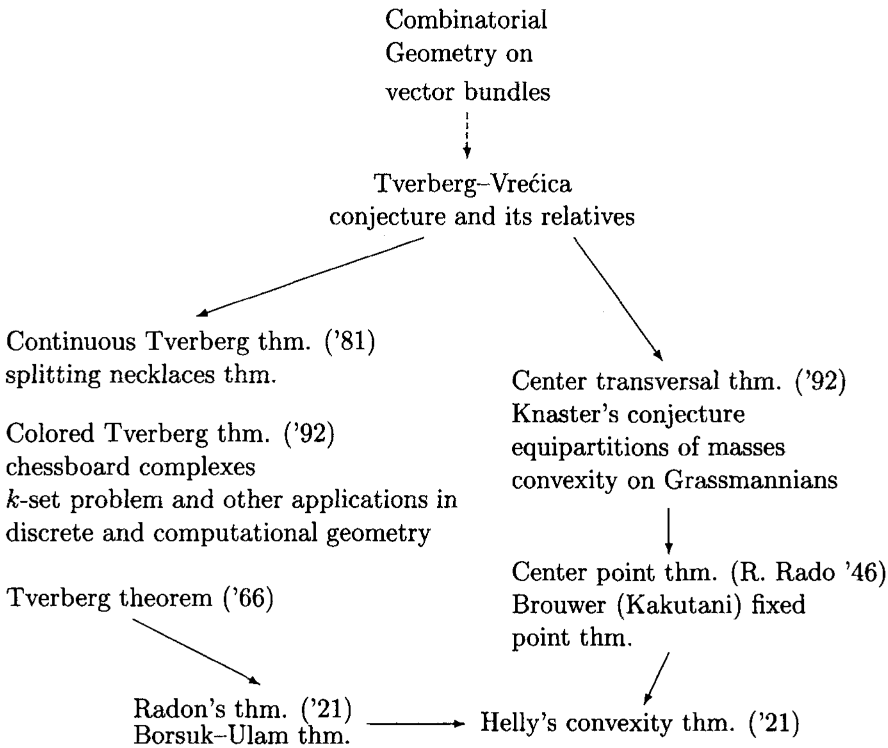

Figure 1.

THEOREM 1.4 (Center transversal theorem, [30]): *Let $\mu_0, \mu_1, \ldots, \mu_k$, $0 \leq k \leq d-1$, be a collection of σ-additive probability measures defined on the σ-algebra of all Borel sets in R^d. Then there exists a k-dimensional affine subspace $P \subseteq R^n$ such that for every closed halfspace $H(v,\alpha) := \{x \in R^n \mid \langle x, v\rangle \leq \alpha\}$ and every $i \in \{1, 2, \ldots, k\}$,*

$$P \subseteq H(v,\alpha) \Longrightarrow \mu_i(H(v,\alpha)) \geq 1/(n-k+1).$$

Conjecture 1.1 reduces to Theorem 1.3 for $k = 0$. Tverberg's theorem (Theorem 1.3) reduces in case $q = 2$ to Radon's theorem, which was the basis for the first proof of Helly's convexity theorem. The continuous Tverberg theorem (Theorem 1.6) is a consequence of the nonlinear version of the Tverberg–Vrećica conjecture which is formulated in section 4. The center transversal theorem follows from Conjecture 1.1 by an approximation argument. This can be done for the class of weak limits of measures concentrated on finite sets. The majority of geometrically interesting measures including the measures absolutely continuous with respect to the Lebesgue measure and the counting measures are obtained this way. Recall that the counting measure ν_S, associated to a finite set $S \subset R^d$, is defined by $\nu_S(A) := |A \cap S|/|S|$.

PROPOSITION 1.5: *The central transversal theorem (Theorem 1.4) is a consequence of the Tverberg–Vrećica conjecture.*

Proof: Suppose that $\mu_i, i = 0, \ldots, k$ is a collection of probability measures such that for each i there exists a sequence q_n^i of natural numbers, $q_n^i \to \infty$, and a sequence $S_n^i \subset R^d$ of finite sets, $|S_n^i| = q_n^i$, so that μ_i is a weak limit of counting probability measures $\nu_n^i := \nu_{S_n}^i$, $\nu_n^i(A) := |A \cap S_n^i|/q_n^i$ for each Borel measurable set A. This means that

$$\lim_{n\to\infty} \int_{R^d} f d\nu_n^i = \int_{R^d} f d\mu_i$$

for every bounded continuous function f, which implies that $\limsup \nu_n^i(F) \leq \mu_i(F)$ for each closed set, say for each closed halfspace. Without loss of generality it can be assumed that the cardinality of the set S_n^i is $q_n^i = (r_n^i - 1)(d-k+1)+1$ for some $r_n^i \in N$. By the Tverberg–Vrećica conjecture, applied on the family $\mathcal{S}_n = \{S_n^i\}_{i=0}^k$, there exists an affine k-dimensional subspace $P_n \subset R^d$ and a partition $S_n^i = \bigcup_{j=1}^{r_n^i} S_n^i(j)$ such that $P_n \cap \operatorname{conv}(S_n^i(j)) \neq \emptyset$ for all i and j. From here we deduce that each closed (and open) halfspace H^+ containing P_n intersects each of the sets $S_n^i(j)$, hence

$$\nu_n^i(H^+) \geq \frac{r_n^i}{q_n^i} \to \frac{1}{d-k+1} \quad \text{for } n \to \infty.$$

The sequence P_n is obviously bounded in the Grassmannian of all affine k-dimensional subspaces of R^d so it can be assumed that it converges to a k-plane P. From here and the inequalities above, it is easily deduced that $\mu_i(H^+) \geq 1/(d-k+1)$ for each halfspace $H^+ \supset P$ and each $i = 0, \ldots, k$. ∎

The center transversal theorem reduces in case $k = 0$ to Rado's theorem "on general measure", [19], which is today better known as the center point theorem, [28]. R. Rado deduced his theorem from Helly's convexity theorem but the converse is also true. Namely, a minimal counterexample to Helly's theorem produces a counting measure which contradicts Rado's theorem. In case $k = d-1$ the center transversal theorem reduces to the "ham sandwich theorem". There are many other intriguing and surprising connections, [8], [18], [28]. Some of these results and connections are recent and some were established more than seventy years ago. It is interesting that a single conjecture can provide a unifying theme for different results, separated by several decades, and this may serve as evidence of the fundamental role of the Tverberg–Vrećica conjecture.

Proofs of statements above are often based on topological ideas, e.g. the center transversal theorem follows from the fact that $0 \neq w_k^{n-k} \in H^{k(n-k)}(G_k(R^n); Z_2)$ where w_k is the top Stiefel–Whitney class of the canonical bundle over the Grassmann manifold $G_k(R^n)$. A topological nature of these statements is exemplified by the following "topological Tverberg theorem" which reduces to the Tverberg theorem above if f is an affine map. Note, however, that topological methods used here require an additional assumption that q is a prime.

THEOREM 1.6 ([4]): *Let q be a prime integer and $\Delta^{(q-1)(d+1)}$ a $(q-1)(d+1)$-dimensional simplex. Then for every continuous map $f\colon \Delta^{(q-1)(d+1)} \to R^d$ there exist disjoint faces $\Delta^{t_1}, \ldots, \Delta^{t_q} \subset \Delta^{(q-1)(d+1)}$ such that $\bigcap_{i=1}^{q} f(\Delta^{t_i}) \neq \emptyset$.*

The proof of Theorem 1.2 will be given in section 4 where it is deduced as a corollary of a general combinatorial statement about continuous cross-sections of vector bundles. The proof involves the whole range of techniques that we now have available in the Combinatorial Geometry of Vector Bundles. They include applications of parametrized, ideal valued index theorems of Dold [7], Fadell and Husseini [9]-[11], Izydorek and Rybicki [13] , Jaworowski [14] and Nakaoka [17] and the combinatorics of deleted joins which was introduced by Sarkaria and which evolved in papers of Sarkaria, Vrećica, Živaljević and others.

It is plausible that a refinement of methods used in the proof of Theorem 1.2 should establish the Tverberg–Vrećica conjecture in the case where the integers r_i are not necessarily equal. Also, we are convinced that the assumption that d and k are odd integers is not essential.

2. A review of geometric combinatorics in R^d

Figure 1 is primarily intended to illustrate the central position of the Tverberg–Vrećica conjecture in a family of well known combinatorial geometric statements in R^d. In the background, this diagram should suggest that there exists a broader perspective on this field. The new ambient space for these results is a bundle E of vector spaces, rather than a single space, or alternatively the space $\Gamma(E)$ of all continuous cross-sections of E instead of R^d. The *geometric combinatorics* of these objects is in this paper referred to as the *combinatorial geometry on vector bundles*. In this section we give a review of some well known "classical" R^d statements as motivating examples for more general "bundle" results.

The well known Kuratowski nonplanarity criterion implies that $K_{3,3}$ is not embeddable in R^2, which implies that for any collection of 3 red and 3 blue points in the plane, there exist two intersecting vertex disjoint line segments with end points of different color. From a collection of 3 blue, 3 white and 3 red points in the plane R^2, one can always select three vertex-disjoint, "rainbow" triangles which have a nonempty intersection. A "rainbow" triangle is a triangle having all vertices of different color. Something similar is possible in the 3-space R^3. This time we need at least 5 points of each color in order to guarantee existence of three vertex disjoint, "rainbow" triangles, which have a nonempty intersection.

These three statements, or their nonlinear analogues, can be abbreviated as follows:

$$(K_{3,3} \to R^2) \Rightarrow (2 \mapsto \text{point}), \tag{1}$$

$$(K_{3,3,3} \xrightarrow{s} R^2) \Rightarrow (3 \mapsto \text{point}), \tag{2}$$

$$(K_{5,5,5} \to R^3) \Rightarrow (3 \mapsto \text{point}). \tag{3}$$

For example, the last statement says that for every continuous map $f\colon K_{5,5,5} \to R^3$, where $K_{5,5,5} := [5]*[5]*[5]$ is the 2-complex obtained as the join of three copies of $[5] = \{1,2,3,4,5\}$, there exist three points in three vertex-disjoint triangles which are mapped to the same point in R^3. The statement (2) is similar, except that $f\colon K_{3,3,3} \to R^2$ is assumed to be a simplicial map and it is not known if it holds in the general case. Recall that the first statement is historically one of the earliest topological results known already to Euler, who formulated it as a problem about three houses and three wells.

In all examples listed above the target space is two or three dimensional space. Actually, these results are special cases of quite general statements about configurations of "colored" points in R^d, see [28], [29], [31]. A different generalization to the 3-space is provided by the theory of *linkless*, *windless* etc. embeddings of graphs, [24]. An example from this circle of results is the statement

$$(K_6 \hookrightarrow R^3) \Rightarrow \text{linking} \tag{4}$$

which says that for every embedding of the graph K_6 in R^3 there exist two disjoint circuits C_1, C_2 of K_6 which are linked with a nonzero winding number, [6], [21].

We show in sections 3 and 4 that the results (1)–(3) above can be extended in a systematic way to include higher dimensional statements where the existence of a common point (common 0-dimensional transversal) is replaced by the existence of a common k-dimensional transversal. Recall that a k-dimensional transversal of a family $\mathcal{F} = \{F_j\}_{j=1}^m$ of subsets in R^d is an affine k-dimensional space $L \subset R^d$ such that $L \cap F_j \neq \emptyset$ for all j. For example, a simple consequence of the "ham sandwich theorem" is the following statement:

$$(K_{4,4} \to R^2) \Rightarrow (4 \mapsto \text{line})$$

which implies that for any collection of 4 black and 4 white points in the plane R^2 there exists a line intersecting four vertex disjoint line segments with end points of different color. Much less trivial is the statement

$$(K_{6,6} \to R^3) \Rightarrow (4 \mapsto \text{line}) \tag{5}$$

which, in the affine case, says that for every collection of 6 red and 6 blue points in R^3 there exist 4 line segments with end points of different color having a common line transversal. This result can be viewed as a relative of the nonplanarity of $K_{3,3}$. Of course, there are higher dimensional complexes which exhibit similar behavior as shown by the following example:

$$(\sigma_2^7 \to R^3) \Rightarrow (4 \mapsto \text{line})$$

where σ_2^7 is the 2-skeleton of a 7-dimensional simplex σ^7. These results will be deduced in the following sections as corollaries of general statements belonging to the combinatorial geometry on vector bundles. This is not a surprise since the collection of all affine k-dimensional subspaces of R^d is naturally interpreted as the total space of the tautological $(d-k)$-dimensional vector bundle over the Grassmannian manifold $G_{d-k}(R^d)$ of all $(d-k)$-dimensional vector subspaces of R^d.

We end this section with an open problem. It is known that aside from planar graphs there exist other topologically defined classes of graphs which admit a combinatorial characterization in terms of "forbidden minors". According to Robertson, Seymour and Thomas, graphs which admit linkless (windless) embeddings can be characterized as graphs which have no minors in the the Petersen family, [24].

PROBLEM 2.1: Find a combinatorial characterization in terms of forbidden minors of all graphs K for which the statement (5) is not true, i.e. characterize all graphs which can be mapped to the 3-space such that no 4 vertex-disjoint edges admit a line transversal.

3. Combinatorics of sections of 2-plane bundles

In this section we prove first a theorem (Theorem 3.1) which is so designed to imply the simplicial version $(K_{6,6} \xrightarrow{s} R^3) \Rightarrow (4 \mapsto \text{line})$ of the statement (5) from the previous section. It also serves as a good illustration of the technique and the main ideas applied in the rest of this paper. After that we formulate a nonlinear "twin" version of Theorem 3.1 and show that its proof doesn't require new ideas. The proof of Theorem 3.1 is carried on step by step, which should hopefully make its ideas more transparent and accessible to nonspecialists in either Combinatorics or Topology.

THEOREM 3.1: *Let $R^2 \to E \xrightarrow{p} M$ be a 2-dimensional, real vector bundle over a 2-manifold M with a nonzero second Stiefel-Whitney class $w_2(E) \in H^2(M; Z_2)$. Let $\mathcal{A} = \{a_\alpha\}_{\alpha=1}^6$ and $\mathcal{B} = \{b_\beta\}_{\beta=1}^6$ be two collections of continuous cross-sections of E. Then for some $p \in M$ there exists a configuration of four intersecting line segments $[a_{\alpha_i}(p), b_{\beta_i}(p)], i = 1, \dots, 4$ formed by these cross-sections. In other words, there is a partial matching $\{(\alpha_i, \beta_i)\}_{i=1}^4 \subset [6] \times [6]$ in the bipartite graph $K_{6,6}$ so that*

$$\bigcap_{i=1}^{4} [a_{\alpha_i}(p), b_{\beta_i}(p)] \neq \emptyset.$$

The plan of the proof of Theorem 3.1 follows.

(i) One observes first that $(K_{3,3} \to R^2) \Rightarrow (2 \mapsto \text{point})$ is a statement of Borsuk-Ulam type. Recall ([5]) that the Borsuk–Ulam theorem says that for each continuous map $f\colon S^d \to R^d$ there exists a pair of (antipodal) points $\{x, -x\}$ such that $f(x) = f(-x)$. The statement (1) is very similar in spirit with the only difference that instead of a pair of antipodal points, it claims the existence of two points x and y, belonging to vertex disjoint edges of $K_{3,3}$, such that $f(x) = f(y)$.

(ii) The second observation is that the Borsuk–Ulam theorem admits a generalization to vector bundles, see [14], [17], [10], [7], [13], and section 5, which is referred to as the parametrized Borsuk-Ulam theorem. We conclude that, in light of (i), the statement (1) also admits a parametrized version.

(iii) Let $\mathcal{A}' := \{a_\alpha\}_{\alpha=1}^3, \mathcal{B}' := \{b_\beta\}_{\beta=1}^3, \mathcal{A}'' := \mathcal{A} \setminus \mathcal{A}'$ and $\mathcal{B}'' := \mathcal{B} \setminus \mathcal{B}'$. Let $\{f_b\}_{b\in M}, f_b\colon K_{3,3} \to E_b$, be the family of simplicial maps determined by sections $\mathcal{A}' \cup \mathcal{B}'$. Define the set C' of **crossing points** for $\mathcal{A}' \cup \mathcal{B}'$ (see Definition 3.3 for a more precise description) by

$$C' := \{x \in E \mid x \text{ is a double point for } f_b\colon K_{3,3} \to E_b \text{ for some } b \in M\}.$$

An application of the parametrized version of (1), coupled with the index theorem (section 5), allows us to conclude that both C' and C'' (the letter set is associated to sections $\mathcal{A}'' \cup \mathcal{B}''$) are (co)homologically "very big".

(iv) Finally, we conclude, essentially by the argument of theorem 11.10 from [5], that $C' \cap C'' \neq \emptyset$, which completes the proof.

Now we are ready to present a reasonably detailed proof of Theorem 3.1 following the plan outlined above.

Proof of Theorem 3.1:

STEP (i): Let us sketch a cohomological proof of nonplanarity of $K_{3,3}$, cf. [23] or [29]. Given $f\colon K_{3,3} \to R^2$ one defines a Z_2-equivariant map $\Phi\colon (K_{3,3})^{*(2)}_{\delta} \to R^2 * R^2, \Phi(tx+(1-t)y) := tf(x)+(1-t)f(y)$ from the deleted join of $K_{3,3} = [3]*[3]$ to $R^2 * R^2 \subset R^5$. Recall ([29]) that the q^{th} deleted join of a simplicial complex K is the complex $K^{*(q)}_{\delta} \subset K * \cdots * K = K^{*(q)}$, which consists of all simplices of the form $\sigma = \sigma_1 * \cdots * \sigma_q$ where $\sigma_i, i = 1, \ldots, q$ are pairwise vertex-disjoint simplices in K. The nonplanarity of $K_{3,3}$ follows from

$$\text{Im}(\Phi) \cap \Delta \neq \emptyset \tag{6}$$

where $\Delta = \{1/2x + 1/2x \in R^2 * R^2 | x \in R^2\}$ is the diagonal in $R^2 * R^2$. Since; [29], $(K_{3,3})^{*(2)}_{\delta} \cong S^3$ and $R^5 \setminus \Delta \simeq S^2$ as Z_2-spaces, the desired observation follows from the fact that there does not exist a Z_2-equivariant map from S^3 to S^2 (the Borsuk-Ulam theorem).

STEP (ii): Suppose that instead of a single map $f\colon S^d \to R^d$, as in the usual Borsuk-Ulam theorem, we have a continuous family of maps $f_b\colon S^d \to R^d, b \in B$, where B is a "parameter" space. In other words, we have a continuous map $F\colon B \times S^d \to B \times R^d$ where $f_b(x) = F(b,x)$. More generally, instead of usual Cartesian products we can use "twisted products" or fibre bundles $S^d \to V \overset{v}{\to} B$ and $R^d \to W \overset{w}{\to} B$ in which case F is a bundle map, i.e. a map for which the following diagram is commutative:

$$\begin{array}{ccc} V & \xrightarrow{F} & W \\ {\scriptstyle v}\downarrow & & \downarrow{\scriptstyle w} \\ B & \xrightarrow{\cong} & B \end{array}$$

The Borsuk-Ulam theorem guarantees that for each $b \in B$ there exists a pair of points $\{x, -x\} \subset V_b := v^{-1}(b)$ such that $F(x) = F(-x)$. The parametrized Borsuk-Ulam theorem, [14], [17], [10], [7], [13], says that we know a great deal more about the set $C := \{x \in V \mid F(x) = F(-x)\} \subset V$ than just $C \cap V_b \neq \emptyset$ for each $b \in B$. Namely, it has been proved that the map $\bar{v}\colon C/Z_2 \to B$ induces a monomorphism

$$\check{H}^*(B, Z_2) \xrightarrow{1-1} \check{H}^*(C/Z_2, Z_2) \tag{7}$$

of Čech cohomology groups, where C/Z_2 is the orbit space with respect to the obvious action of Z_2.

Remark 3.2: The reader who prefers to work with singular cohomology groups should replace C/Z_2, in the statement above, by an arbitrary (small) open neighborhood $O(C/Z_2)$ of C/Z_2. This change wouldn't affect any of the subsequent proofs. Our choice of the Čech cohomology keeps the notation simpler and often, like in the case of index theorems (section 5), leads to aesthetically more satisfying formulas.

Something similar to (7) ought to be true for other Borsuk–Ulam type results so, in light of (i), the statement (1) should also have its parametrized version. This is established in our next step.

STEP (iii): Cross-sections $\mathcal{A}' \cup \mathcal{B}'$ (see (iii) above) define, by simplicial extension, a map $F\colon M \times K_{3,3} \to E$ of two M-fiber bundles. As in the proof of nonplanarity of $K_{3,3}$ (cf. (i)) it is natural to pass to the Z_2-equivariant map Φ of bundles over M

$$\begin{array}{ccc} M \times S^3 & \xrightarrow{\Phi} & W \\ \downarrow & & \downarrow \\ M & \xrightarrow{\cong} & M \end{array}$$

where $S^3 \cong (K_{3,3})^{*(2)}_{\delta}$ and W is the bundle with the fibre $W_p = E_p * E_p$. The last bundle is realized as a subbundle of the Whitney sum $Z := E \oplus \theta^1 \oplus E$, where θ^1 is a 1-dimensional trivial bundle over M in the same way $R^2 * R^2$ can be realized inside R^5, [29]. The original bundle E is naturally isomorphic with the diagonal subbundle $\Delta \to M$ of W where $\Delta_p = \{1/2x + 1/2x \in W_p | x \in E_p\}$, and we assume until the end of this proof that bundles E and Δ are identified.

Definition 3.3 (crossing points): It follows from nonplanarity of $K_{3,3}$ that for every $p \in M$ there exists a partial matching $\{(\alpha_i, \beta_i)\}_{i=1}^2$ in the graph $K_{3,3}$ so that $[a_{\alpha_1}(p), b_{\beta_1}(p)] \cap [a_{\alpha_2}(p), b_{\beta_2}(p)] \neq \emptyset$. A point in this intersection will be called a *crossing* point, or briefly a C-point. Let $C' := \{x \in E \mid x \text{ is a } C\text{-point in } E_p \text{ for some } p \in M\}$ be the set of all crossing points for sections $\mathcal{A}' \cup \mathcal{B}'$ and, similarly, let C'' be the set of all crossing points for sections $\mathcal{A}'' \cup \mathcal{B}''$. In light of the identification of bundles E and Δ, we observe that

$$C' = \mathrm{Im}(\Phi) \cap \Delta.$$

Note that the number of C-points can vary from fiber to fiber, so the restriction π_0: $C' \to M$ of π: $E \to M$ is far from being a covering map. It was already observed that π_0 is onto. We want to show that C' is a "cohomologically big" space in the sense that π_0^*: $H^*(M; Z_2) \to H^*(C'; Z_2)$ is a monomorphism (cf. Remark 3.2). This is deduced as follows.

A very useful invariant of a G-space X is the G-index $\mathrm{Ind}_G(X)$; see section 5. If X is a family of G-spaces parametrized by a parameter space M, then the index $\mathrm{Ind}_G(X)$ is an ideal in the ring $H^*(M; R) \otimes_R H^*(BG; R)$, provided the conditions of the Künneth theorem are satisfied. Let $G = Z_2 = R$. By applying the parametrized version of the index theorem to the map Φ above, see section 5 or the references [7], [10], [13], we have

$$\mathrm{Ind}_G(\Phi^{-1}(\Delta))\,\mathrm{Ind}_G(W \setminus \Delta) \subset \mathrm{Ind}_G(M \times S^3). \tag{8}$$

The ring $H^*(M) \otimes_{Z_2} H^*(BZ_2) \cong H^*(M)[\Omega], \deg(\Omega) = 1$, is a polynomial ring with coefficients in $H^*(M)$. The relation (8) has in this case a very simple meaning. The index $\mathrm{Ind}_G(S(V))$ of the sphere bundle associated to a vector bundle V, cf. Proposition 5.4, is the principal ideal in the ring $H^*(M)[\Omega]$ generated by a polynomial of the form $\Omega^k + w_1\Omega^{k-1} + \cdots + w_k$, where w_i are the Stiefel–Whitney classes of V and $k = \dim(V)$. As a consequence we obtain that the ideal $\mathrm{Ind}_G(M \times S^3)$ is generated by the polynomial Ω^4, the index $\mathrm{Ind}_G(W \setminus \Delta)$ is generated by a polynomial of the form $\Omega^3 + \gamma_1\Omega^2 + \gamma_2\Omega + \gamma_3$ and the relation (8) says that

$$\theta(\Omega^3 + \gamma_1\Omega^2 + \gamma_2\Omega + \gamma_3) \quad \text{is divisible by } \Omega^4 \tag{9}$$

for each $\theta \in \mathrm{Ind}_G(\Phi^{-1}(\Delta))$. This implies that there does not exist a constant polynomial θ in the index $\mathrm{Ind}_G(\Phi^{-1}(\Delta))$, where constant means a polynomial of Ω-degree 0. From here we deduce that $H^*(M) \to H^*(\Phi^{-1}(\Delta)/Z_2)$ is a monomorphism. Finally, from the commutative diagram

$$\begin{array}{ccc} \Phi^{-1}(\Delta)/Z_2 & \longrightarrow & C' \\ \downarrow & & \downarrow{\scriptstyle \pi_0} \\ M & \longrightarrow & M \end{array}$$

it follows that π_0^* is also a monomorphism.

STEP (iv): By the universal coefficients theorem

$$\mathrm{Hom}(H_2(M;Z_2),Z_2) \longrightarrow \mathrm{Hom}(H_2(C';Z_2),Z_2)$$

is 1-1, so $(\pi_0)_*\colon H_2(C';Z_2) \to H_2(M;Z_2)$ is an epimorphism. Hence, the Z_2-fundamental class $e \in H_2(M;Z_2) \cong H_2(E;Z_2)$ is supported by C'. If $D(E)$ and $S(E)$ are the associated disk and sphere bundles, we conclude that the Thom class $\tau \in H^2(D(E),S(E);Z_2)$, seen as the dual of e in the manifold with boundary $(D(E),S(E))$ ([5]), is supported by an arbitrary small neighborhood of C'. Everything that has been proved for C' also holds for the set C'' of crossing points associated to $\mathcal{A}'' \cup \mathcal{B}''$. Since $\tau^2 = w_2\tau \in H^*(D(E),S(E);Z_2)$ is nonzero and τ is supported by arbitrary small neighborhoods of both C' and C'', we conclude, [5] Theorem 11.10, that $C' \cap C'' \neq \emptyset$ and the theorem follows. ∎

Here we formulate a stronger, nonlinear version of Theorem 3.1 which nevertheless doesn't need new ideas for its proof.

THEOREM 3.4: *Let $R^2 \to E \xrightarrow{\pi} M$ be a 2-dimensional, real vector bundle over a 2-manifold M with a nonzero second Stiefel–Whitney class $w_2(E) \in H^2(M;Z_2)$. Let $\phi_p\colon K_{6,6} \to E$ be a family of maps continuously depending on the parameter $p \in M$ such that $\mathrm{Image}(\phi_p) \subset E_p := \pi^{-1}(p)$. Then there exists $p \in M$ and four vertex-disjoint edges $e_1,\ldots,e_4$ of the graph $K_{6,6}$ such that*

$$\bigcap_{i=1}^{4} \phi_p(e_i) \neq \emptyset.$$

Proof: The proof is completely analogous to the proof of Theorem 3.1 with the only difference that in the step (iii), the bundle map $\Phi\colon M \times (K_{3,3})^{*(2)}_{\delta} \to W$ is defined directly with the aid of functions ϕ_p by $\Phi(p,tx+(1-t)y) = t\phi_p(x) + (1-t)\phi_p(y)$. ∎

COROLLARY 3.5: *For every continuous map $f\colon K_{6,6} \to R^3$ there exist four vertex-disjoint edges $e_1,\ldots,e_4$ in the graph $K_{6,6}$ such that the sets $f(e_1),\ldots,f(e_4)$ have a common line-transversal in R^3 or, in the notation of section 2,*

$$(K_{6,6} \to R^3) \Rightarrow (4 \mapsto \text{line}).$$

Proof: The space of all lines in R^3 is identified as the tautological 2-plane bundle E over the projective space RP^2. More explicitly, a line $l \subset R^3$ is represented by the vector $v := l \cap p$ in the fibre p, where $p := l^{\perp}$ is the plane orthogonal to l which passes through the origin. The result follows from Theorem 3.4 if we define $\phi_p := \pi_p \circ f$, where $\pi_p\colon R^3 \to p$ is the orthogonal projection. ∎

4. T–V conjecture and its relatives

Here we present proofs of the main results of this paper including a partial solution of the Tverberg–Vrećica conjecture. We use the parametrized, cohomological index theory as one of our main tools; see the references or section 5. We suppose that our cohomology theory is a continuous extension of the singular cohomology theory, say Alexander–Spanier or Čech theory. In particular, we assume that the cohomology of each closed subspace T in a compact manifold (with boundary) is a direct limit of singular cohomology groups taken over all open neighborhoods of T,

$$H^*(T) \cong \mathbf{colim}\{H^*_s(U) \mid U \text{ is an open neighborhood of } T\}.$$

This means that the reader who prefers to work with the singular theory should not have difficulties (see Remark 3.2) to make necessary modifications.

Definition 4.1: Let $R^d \to E \xrightarrow{\pi} B$ be a d-dimensional, real vector bundle over a compact space and $\mathcal{G} = \{\gamma^i\}_{i\in I}, I = \{0, 1, \dots, (q-1)(d+1)\}$, a collection of continuous cross-sections where q is an odd prime number. Then $x \in E$ is called a *Tverberg* q-point or simply a t_q-point if there exists $b \in B$ and a partition $I = I_1 \cup \dots \cup I_q$ of I into q nonempty, pairwise disjoint sets so that

$$x \in \bigcap\{\mathrm{conv}\{\gamma^k(b)\}_{k\in I_j} \mid j = 1, \dots, q\} \subset E_b := \pi^{-1}(b).$$

THEOREM 4.2: *Let $R^d \to E \xrightarrow{\pi} B$ be a d-dimensional, real vector bundle over a compact space B and $\mathcal{G} = \{\gamma^i\}_{i\in I}, I = \{0, 1, \dots, (q-1)(d+1)\}$, a collection of continuous cross-sections where q is an odd prime number. Let $T \subset E$ be the collection of t_q-points. Under these conditions the restriction map $\pi_0\colon T \to B$ induces a 1-1 map*

$$\pi_0^*\colon H^*(B, Z_q) \longrightarrow H^*(T, Z_q). \tag{10}$$

Proof: Let Δ^Λ be a Λ-dimensional simplex, $\Lambda = (q-1)(d+1)$. Let K be the configuration space of all q-tuples of points $x_1, \dots, x_q \in \Delta^\Lambda$ such that for all $i \neq j$, x_i and x_j belong to disjoint faces of Δ^Λ. K is easily seen to be a regular cell subcomplex of $(\Delta^\Lambda)^q$ with the maximal cells of the form $e_{\mathcal{I}} = \Delta^{I_1} \times \dots \times \Delta^{I_q}$, where $\mathcal{I} = \{I_1, \dots, I_q\}$ is a partition of I and Δ^{I_j} the corresponding faces of Δ^Λ. Note that K is a free Z_q-space with the action inherited from the obvious Z_q-action on $(\Delta^\Lambda)^q$. This complex was introduced in [4] where it was used in a proof of the topological Tverberg theorem (Theorem 1.6). A key property of this space is that it is a $[(q-1)d-1]$-connected, $[(q-1)d]$-dimensional CW-complex.

For each $b \in B$ there is a simplicial map $A_b\colon \Delta^\Lambda \to E_b$, continuously depending on $b \in B$, which sends vertices of Δ^Λ to vectors $\gamma^i(b), i = 0, \ldots, \Lambda$. Let $F_b\colon K \to E_b^{\oplus q}$ be the induced map into the direct sum of q-copies of E_b. These maps also continuously depend on b and, taken together, define a map

$$B \times K \xrightarrow{F} E^{\oplus q}.$$

F is viewed as a Z_q-equivariant map of two Z_q-spaces parametrized by B. Let Δ be the diagonal subbundle of $E^{\oplus q}$. By the parametrized index theorem (see the end of section 5) applied on the Z_q-equivariant map F with the subspace Δ in the role of V, we have

$$\text{(11)} \qquad \operatorname{Ind}(F^{-1}(\Delta)) \operatorname{Ind}(E^{\oplus q} \setminus \Delta) \subset \operatorname{Ind}(B \times K).$$

The bundle $E^{\oplus q} \setminus \Delta$ has the homotopy type of a free $[(q-1)d-1]$-dimensional Z_q-sphere bundle over B. By a Z_q-extension of Proposition 5.4, see [13], $\operatorname{Ind}(E^{\oplus q} \setminus \Delta)$ is an ideal in the ring $R = H^*(B, Z_q) \otimes Z_q[x, y]$ where $\deg(x) = 1, \deg(y) = 2$. More precisely, $\operatorname{Ind}(E^{\oplus q} \setminus \Delta)$ generated by a polynomial of the form

$$\text{(12)} \qquad Q = y^n + \alpha_1 x y^{n-1} + \beta_1 y^{n-1} + \cdots + \alpha_n x + \beta_n$$

where $n = (q-1)d/2$ and $\alpha_i, \beta_i \in H^*(B, Z_q), i = 1, \ldots, n$ are the so-called Grothendieck–Chern classes of the bundle $E^{\oplus q} \setminus \Delta \to B$, see [7], [13]. Since K is $[(q-1)d-1]$-connected we conclude that the map $l^*\colon H^*(BZ_q, Z_q) \to H^*(K/Z_q, Z_q)$, induced by the classifying map $l\colon K/Z_q \to BZ_q$, is 1-1 up to dimension $(q-1)d$. From here and the universal coefficients theorem it follows that the composite map

$$L\colon H^*(B) \otimes Z_q[x, y] \longrightarrow H^*(B) \otimes H^*(K/Z_q) \xrightarrow{1-1} H^*(B \times K/Z_q)$$

has the property that $L^*(Q) \neq 0$ for any nonzero polynomial Q of degree at most $(q-1)d$. In other words, the ideal $\operatorname{Ind}(B \times K)$ does not contain a nonzero polynomial Q of degree $\deg(Q) \leq (q-1)d$. This immediately implies that for any $0 \neq \alpha \in H^*(B, Z_q), \alpha = \alpha \otimes 1 \notin \operatorname{Ind}(F^{-1}(\Delta))$. Otherwise, by the index theorem, the polynomial $P = \alpha Q$ would be a nonzero element in $\operatorname{Ind}(B \times K)$ of degree at most $(q-1)d$. Since the map $p\colon F^{-1}(\Delta)/Z_q \to B$ can be factored as $p = \pi_0 \circ F$, we conclude that $\pi_0^*\colon H^*(B, Z_q) \to H^*(T, Z_q)$ is also injective. ∎

Remark 4.3: Under the same conditions, using the universal coefficients theorem for cohomology, we deduce that the map $\pi_0^*\colon H^*(B, Z) \otimes Z_q \to H^*(T, Z) \otimes Z_q$ is also injective.

Remark 4.4: If we work with singular cohomology theory, an analogue of Theorem 4.2 claims that

$$\pi_0^*\colon H^*(B, Z_q) \longrightarrow H^*(V, Z_q) \tag{13}$$

is a monomorphism for arbitrary small open neighborhood V of the space T of Tverberg q-points.

Remark 4.5: Theorem 4.2 was modeled by the usual Tverberg theorem. It is clear that the method can be used quite generally for other results of this type. For example, colored Tverberg theorems ([31], [27]), specially the form given in [27], is suitable for such an extension. The requirement in Theorem 4.2 that q is a prime is essential for the methods used here, but one can easily modify the proof above to include the case $q = 2$.

COROLLARY 4.6: *Under the assumptions of Theorem 4.2 it follows that the map*

$$(\pi_0)_*\colon H_*(V, Z_q) \longrightarrow H_*(B, Z_q)$$

of singular homology groups, induced by the projection $\pi_0\colon V \to B$, is an epimorphism for each open neighborhood V of T. Specially if B is a Z_q-orientable, n-dimensional manifold and $[B] \in H_n(B, Z_q)$ is the corresponding fundamental class, then $[B] \in H_n(V, Z_q)$. In other words, $[B]$, seen as a class in $H_n(E, Z_q) \cong H_n(B, Z_q)$, is represented by a cycle in V.

Proof: Suppose that $i\colon T \to E$ is the inclusion map. Then there is a commutative square

$$\begin{array}{ccc} H^*(T, Z_q) & \xleftarrow{\ i^*\ } & H^*(E, Z_q) \\ \uparrow{\scriptstyle \pi_0^*} & & \uparrow{\scriptstyle \pi^*} \\ H^*(B, Z_q) & \xleftarrow{\ \cong\ } & H^*(B, Z_q) \end{array} \tag{14}$$

From this diagram, since π^* is an isomorphism, it follows that $H^*(E, Z_q) \xrightarrow{i^*} H^*(T, Z_q)$ is a monomorphism. From the inclusions $T \subset V \subset E$, where V is an arbitrary open neighborhood of T, we observe that $H^*(E, Z_q) \to H^*(V, Z_q)$ is also a monomorphism where (Remark 4.4) we can assume that these are the singular cohomology groups. By the Universal coefficients theorem for singular cohomology with coefficients in Z_q, where q is assumed to be a prime, $H^*(X, Z_q) \cong \mathrm{Hom}(H_*(X, Z_q), Z_q)$. We conclude that the map $H_*(V, Z_q) \to H_*(E, Z_q)$ of singular homology groups, induced by the inclusion $V \subset E$, is an epimorphism and the corollary follows. ∎

CONJECTURE 4.7 (Grassmannian Tverberg–Vrećica conjecture): *Let*

$$\mathcal{G} = \{\Gamma^{\alpha\beta} \mid 0 \le \alpha \le k, 0 \le \beta \le (r_\alpha - 1)(d-k+1)+1\} = \bigcup_{\alpha=0}^{k} \mathcal{G}_\alpha$$

be a collection of continuous cross-sections of the canonical $(d-k)$-dimensional bundle over the Grassmann manifold $G_{d-k}(R^d)$ of all $(d-k)$-subspaces of R^d. We assume that $r_\alpha \ge 2, \alpha = 0,\ldots,k$ are arbitrary integers. Then there exists $p \in G_{d-k}(R^d)$ and a partition $I_\alpha^1 \cup \cdots \cup I_\alpha^{r_\alpha}$ of $\{0,1,\ldots,(r_\alpha - 1)(d-k+1)\}$ for each $\alpha = 0,\ldots,k$, so that

$$\bigcap_{\alpha=0}^{k} \bigcap_{j=1}^{r_\alpha} \mathrm{conv}\{\Gamma^{\alpha\nu}(p)\}_{\nu\in I_\alpha^j} \ne \emptyset.$$

It is not difficult to see that the original Tverberg–Vrećica conjecture follows from Conjecture 4.7. Our Theorem 1.2 is a consequence of the following result.

THEOREM 4.8: *Conjecture 4.7 is true under the condition that both d and k are odd integers and that natural numbers $r_\alpha, \alpha = 0,\ldots,k$ are all equal to an odd prime number r.*

The following proposition will be needed in the proof of Theorem 4.8.

PROPOSITION 4.9: *Let $E \to G_{2k}^+(R^d)$ be the canonical $(2k)$-dimensional vector bundle over the Grassmann manifold $G_{2k}^+(R^d)$ of all oriented $(2k)$-dimensional vector subspaces of R^d. Assume that $d \ge 3$ is an odd integer. Then the Euler class $e := e(E) \in H^{2k}(G_{2k}^+(R^d), Z_r)$ has the property $e^{d-2k} \ne 0$ for any coefficient ring Z_r, where r is an odd integer.*

Proof: The proof uses the ideas of the proof of Theorem 3.16 in [10]. Denote by $V_{d,2k}$ the Stiefel manifold of all orthonormal $(2k)$-frames in R^d and let $T(j) = \mathrm{SO}(2) \times \cdots \times \mathrm{SO}(2)$ be the product of j copies of SO(2). Let us collect first some useful information about the cohomology of $G_2^+(R^d)$ with Z_r-coefficients. From the fibration $S^{d-2} \to V_{d,2} \to S^{d-1}$ one computes the cohomology of $V_{d,2}$. From here, by using the fibration $\mathrm{SO}(2) \to V_{d,2} \to G_2^+(R^d)$, one deduces that $H^\bullet(G_2^+(R^d), Z_r) \cong Z_r[x]/(x^{d-1})$, where $x \in H^2(G_2^+(R^d), Z_r)$ is the Euler class of the canonical 2-bundle over $G_2^+(R^d)$. Note that this description is a consequence of the fact that $2 \in Z_r$ is invertible if r is an odd integer. The proof of the desired result $e^{d-2k} \ne 0$ is by induction. We start with the fibration

$$V_{d-2k+2,2} \xrightarrow{\ i\ } V_{d,2k} \xrightarrow{\ \pi\ } V_{d,2k-2} \tag{15}$$

where π is the map which forgets the last two vectors in an orthonormal frame $o \in V_{d,2k}$. We have a natural group action of $T(k)$ on $V_{d,2k}$ and the exact sequence

$$\mathrm{SO}(2) \longrightarrow T(k) \longrightarrow T(k-1)$$

of groups shows how the torus $T(k)$ acts on the fibration (15). Dividing by this action we get the fibration

$$(16) \qquad \tilde{V}_{d-2k+2,2} \xrightarrow{\tilde{i}} \tilde{V}_{d,2k} \xrightarrow{\tilde{\pi}} \tilde{V}_{d,2k-2}.$$

The space $\tilde{V}_{d,2k} := V_{d,2k}/T(k)$ is the base space of the principal $T(k)$ bundle

$$(17) \qquad T(k) \longrightarrow V_{d,2k} \longrightarrow \tilde{V}_{d,2k}$$

so let $\tilde{V}_{d,2k} \xrightarrow{p_{d,k}} \mathrm{B}(T(k))$ be the classifying map. The cohomology of $\mathrm{B}(T(k))$ is a graded polynomial algebra $Z[z_1, z_2, \ldots, z_k], \deg(z_i) = 2$, where z_i is the generator of the i^{th} component in the decomposition $\mathrm{B}(T(k)) = \mathrm{BSO}(2) \times \cdots \times \mathrm{BSO}(2)$. We want to show that the class $w_{n,k} = (z_1 z_2 \cdots z_k)^{d-2k}$ is not in the kernel of the map $p^*_{d,k}$. This will be proved by induction on k. For $k = 1$, the observation is true because of the description of $H^*(G_2^+(R^d); Z_r)$ given above. For the inductive step, let us collect all needed fibrations in a commutative diagram

$$(18) \qquad \begin{array}{ccccc}
V_2(R^{d-2k+2}) & \longrightarrow & G_2^+(R^{d-2k+2}) & \xrightarrow{p} & \mathrm{B}T(1) \\
\downarrow{\scriptstyle i} & & \downarrow{\scriptstyle \tilde{i}} & & \downarrow \\
V_{d,2k} & \longrightarrow & \tilde{V}_{d,2k} & \xrightarrow{p_{d,k}} & \mathrm{B}T(k) \\
\downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \tilde{\pi}} & & \downarrow \\
V_{d,2k-2} & \longrightarrow & \tilde{V}_{d,2k} & \xrightarrow{p_{d,k-1}} & \mathrm{B}T(k-1)
\end{array}$$

where $p := p_{d-2k+2,1}$. Since $p^*\colon H^*(\mathrm{B}T(1)) \to H^*(G_2^+(R^{d-2k+2}))$ is an epimorphism we observe that the conditions of the Leray Hirsch theorem are satisfied for the fibration $G_2^+(R^{d-2k+2}) \to \tilde{V}_{d,2k} \to \tilde{V}_{d,2k-2}$. By the inductive hypothesis, the image of $(z_1 z_2 \cdots z_{k-1})^{d-2k+2}$ is nonzero in $H^*(\tilde{V}_{d,2k-2}, Z_r)$. By the Leray–Hirsch theorem, the image of $(z_1 z_2 \cdots z_{k-1})^{d-2k+2} z_k^{d-2k}$ in $H^*(\tilde{V}_{d,2k}, Z_r)$ is also nonzero, specially $(z_1 z_2 \cdots z_{k-1} z_k)^{d-2k}$ is the nonzero element of the same group. The following diagram is commutative up to homotopy:

$$\begin{array}{ccc}
\tilde{V}_{d,2k} & \xrightarrow{p_{d,k}} & \mathrm{B}T(k) \\
{\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \mu} \\
G_{2k}^+(R^d) & \longrightarrow & \mathrm{BSO}(2k)
\end{array}$$

The map $\nu\colon G^+_{2k}(R^d) \to \mathrm{BSO}(2k) \cong G^+_{2k}(R^\infty)$ classifies the canonical $(2k)$-bundle over $G^+_{2k}(R^d)$ and the Euler class $e(E) \in H^{2k}(G^+_{2k}(R^d), Z_r)$ is the ν^*-image of the Euler class ϵ of the universal bundle over $\mathrm{BSO}(2k)$. From here and the fact that the $p^*_{d,k}$-image of $(z_1 \cdots z_k)^{d-2k}$ is nonzero, we finally deduce that $e^{d-2k} \neq 0$ since

$$s^*(e^{d-2k}) = s^*\nu^*(\epsilon) = p^*_{d,k}((z_1 z_2 \cdots z_k)^{d-2k}) \neq 0.$$

Proof of Theorem 4.8: Now we are ready to give a proof of Theorem 4.8. We will actually prove a little more than promised by showing that a strengthened version of the conjecture is true. Namely, we show that the Grassmann manifold $G_{d-k}(R^d)$ of all $(d-k)$-subspaces in Theorem 4.8 can be replaced by the Grassmann manifold $G^+_{d-k}(R^d)$ of all oriented $(d-k)$-subspaces of R^d. Clearly, a family of cross-sections of the canonical bundle over $G_{d-k}(R^d)$ leads to a family of cross-sections of the canonical $(d-k)$-bundle over $G^+_{d-k}(R^d)$ and it suffices to establish the conjecture in this case. We start with the observation that each of the collections $\mathcal{G}_\alpha = \{\Gamma^{\alpha\beta}\}_{\beta\in I}$, where $I := \{0, \dots, (r-1)(d-k+1)\}$, satisfies the conditions of Theorem 4.2. Let $T_\alpha \subset E$ be the corresponding set of t_r-points (Definition 4.1). Then by Theorem 4.2 (Remark 4.4) and Corollary 4.6, the fundamental class $[B] \in H_{k(d-k)}(G^+_{d-k}(R^d), Z_r)$ is supported by each of the sets $T_\alpha, \alpha = 0, \dots, k$. Let $D(E)$ and $S(E)$ be the disc and sphere bundles associated to E so that $T_\alpha \subset D(E) \setminus S(E)$ for all α. Then $D(E)$, viewed as a manifold with boundary, is orientable since both the base manifold $G^+_{d-k}(R^d)$ and the bundle E are orientable. Hence the Thom class $\tau \in H^{d-k}(D(E), S(E); Z_r)$, as a Poincaré–Lefschetz dual of the fundamental class $[B] \in H_{k(d-k)}(D(E), Z_r)$, is supported by an arbitrary small neighborhood U_α of the set $T_\alpha, \alpha = 0, \dots, k$. Precisely, this means that τ is in the image of the map

$$H^{d-k}(D(E), D(E) \setminus U_\alpha; Z_r) \longrightarrow H^{d-k}(D(E), S(E); Z_r).$$

If the theorem is not true, then $\bigcap_{\alpha=0}^k T_\alpha = \emptyset$ and there exist open sets $U_\alpha \supset T_\alpha$ such that $\bigcap_{\alpha=0}^k U_\alpha = \emptyset$, i.e. $\bigcup_{\alpha=0}^k (D(E) \setminus U_\alpha) = D(E)$. From here we deduce that $\tau^{k+1} = 0$, since this class must be in the image of the map

$$H^{(k+1)(d-k)}(D(E), D(E); Z_r) \longrightarrow H^{(k+1)(d-k)}(B(E), S(E); Z_r).$$

On the other hand, $\tau^{k+1} = \tau e^k$, where $e = e(E)$ is the Euler class of the bundle E, and by the Thom isomorphism theorem and Proposition 4.9 we conclude that $\tau^{k+1} \neq 0$. This contradiction proves the theorem. ∎

Theorem 4.8 has a nonlinear generalization which relates to Theorem 4.8 in the same way as Theorem 3.4 refers to Theorem 3.1. Also, many other Tverberg-type results listed in Figure 1 have their linear and nonlinear extensions to vector bundles, cf. Remark 4.5. We hope to return to these and other related questions in a subsequent paper.

5. A review of index theory

An **index function** is a functor Ind: G-Top $\to \mathcal{C}$ from a category of G-spaces and G-equivariant maps to a small category $\mathcal{C}$. The object $a_X = \mathrm{Ind}(X) \in \mathcal{C}$, associated to a space $X \in$ G-Top, is interpreted as a $\mathcal{C}$-valued degree of complexity of X. The category $\mathcal{C}$ is usually a partially ordered set $(P, \preceq)$ viewed as a small category with $\mathrm{Ob}(\mathcal{C}) = P$ and $\mathrm{Mor}(\mathcal{C}) = \{(p,q) \in P^2 \mid p \preceq q\}$. In other words, this category has a unique morphism $p \to q$ for each pair $p, q \in P$ such that $p \preceq q$. We say that a space X is of lower G-complexity than a space Y if $a_X \preceq a_Y$ and the Borsuk–Ulam paradigm can be formulated as the statement that only spaces of lower complexity can be equivariantly mapped to spaces of higher complexity,

$$\mathrm{Hom}_G(X, Y) \neq \emptyset \quad \Longrightarrow \quad a_X \preceq a_Y.$$

The simplest and historically first examples of index functions were defined for the category of spaces with continuous fixed point free involutions and with the poset N (or $N \cup \{+\infty\}$) of natural numbers as the small category of complexities. An example is the Yang index function Ind^Y defined for free (paracompact) Z_2-spaces X as the smallest number n for which there exists a Z_2-equivariant map from X to the sphere S^n. Yang formulated the following **index theorem.**

THEOREM 5.1: *Let X be a paracompact space, T: $X \to X$ a continuous involution without fixed points (a free Z_2-action) and f: $X \to R^n$ a map such that $f(Tx) = -f(x)$ (a Z_2-equivariant map). Then*

$$\mathrm{Ind}^Y(f^{-1}(0)) \geq \mathrm{Ind}^Y(X) - n.$$

Conner and Floyd defined a closely related index function $\mathrm{Ind}^{CF}(X) := \sup\{n \mid w_X^n \neq 0\}$, where $w_X \in H^1(X/Z_2; Z_2)$ is the first Stiefel–Whitney class of the line bundle $X \times_{Z_2} R^1 \to X/Z_2$. The Yang index function can be easily extended to the case of any finite group G; see, e.g., [29] for an elementary exposition. This index function is denoted by $\mathrm{Ind}_G^Y(X)$, or simply by $\mathrm{Ind}^Y(X)$ if the group G is self-evident from the context. Some combinatorial ideas of K. Sarkaria, [22], [23], see also [15], can be rephrased in the form of the following index theorem which was formulated in [29]. Recall that the order complex $\Delta(P)$ of a poset P is the simplicial complex consisting of all chains in P.

THEOREM 5.2: *Let K be a finite, free G-simplicial complex and Ind_G^Y an appropriate extension of the Yang index function to arbitrary finite groups. Let L be a G-invariant subcomplex. Let P_K be the associated poset $(K, \subset)$, P_L the subposet associated to L and $Q_L := P_K \setminus P_L$ the complementary poset. Let $\Delta(Q_L)$ be the order complex of Q_L. Then*

$$\mathrm{Ind}_G^Y(L) \geq \mathrm{Ind}_G^Y(K) - \mathrm{Ind}_G^Y(\Delta(Q_L)) - 1.$$

Fadell and Husseini, [9], [10], [11], motivated partially by questions related to the extension of the Ljusternik–Schnirelmann method in critical point theory, were led to the ideal-valued cohomological index Ind_G^{FH} : G-Top $\to \mathcal{P}_G$ where $(\mathcal{P}_G, \supset)$ is the poset of all ideals in the ring $H^*(BG; R)$ ordered by the reversed inclusion. BG is as usual the classifying space of G and R a ring of coefficients. For a G-space X let $X_G := EG \times_G X$ be the corresponding homotopic orbit space, where EG is a universal G-space. Then there exists a homotopically unique classifying map $\pi_X \colon X_G \to BG$, hence a precisely and functorially defined homomorphism $\pi_X^* \colon H^*(BG, R) \to H^*(X_G, R)$. Let $\mathrm{Ind}_G^{FH}(X) = \mathrm{Ker}(\pi_X^*)$ be the corresponding ideal in the ring $H^*(BG, R)$. Then if $f \colon X \to Y$ is a G-equivariant map, there exists a commutative diagram

$$\begin{array}{ccc} H^*(Y_G, R) & \longrightarrow & H^*(X_G, R) \\ \uparrow{\scriptstyle \pi_Y^*} & & \uparrow{\scriptstyle \pi_X^*} \\ H^*(BG, R) & \xrightarrow{\cong} & H^*(BG, R) \end{array}$$

which implies that $\mathrm{Ind}_G^{FH}(X) \supset \mathrm{Ind}_G^{FH}(Y)$. An analogue of the Yang index theorem is the following result, [10], where the ideals are multiplied in the usual way:

THEOREM 5.3: *Let $\Phi \colon X \to Y$ be a G-equivariant map and $V \subset Y$ a G-invariant subset. Then*

$$\mathrm{Ind}_G^{FH}(\Phi^{-1}(V))\, \mathrm{Ind}_G^{FH}(Y \setminus V) \subset \mathrm{Ind}_G^{FH}(X).$$

It is shown in [10] that an index theorem of the above form can be deduced in a quite general situation under the condition that the index functor Ind_G satisfies three basic properties or axioms. These axioms can be rephrased as the *monotonicity, additivity* and *continuity* of the index function. A typical category suitable for such an index function is a category of G-spaces over a fixed "parameter" space P. An object in this category is a G-equivariant map $\alpha_X \colon X \to P$, where the action of G on P is assumed to be trivial, while morphisms are the corresponding commutative diagrams. Since the homotopic orbit space X_G is now viewed as a bundle over both the parameter space P and the classifying space BG, we are led to the following definition of the index:

$$\mathrm{Ind}(X) = \mathrm{Ind}_G^P(X) := \mathrm{Ker}\{H^*(P \times BG, R) \longrightarrow H^*(X_G, R)\} \subset H^*(P \times BG, R).$$

The subscript G and the superscript P are often omitted if both the group G and the parameter space P are clear from the context. It is not difficult to show that this index function satisfies all axioms above for a suitable continuous extension of the singular cohomology theory. Hence an index theorem completely analogous to Theorem 5.3 also holds. If the cohomological structure of the parameter space P permits us to use the Künneth formula, then

$$H^*(P \times BG, R) \cong H^*(P, R) \otimes H^*(BG, R)$$

and in some cases this ring can be described as a graded $H^*(P, R)$-algebra. Actually, thanks to the Künneth formula for cohomology which implies that the homomorphism

$$H^*(P) \otimes H^*(BG) \longrightarrow H^*(P \times BG)$$

is always injective, we can compute indices in $H^*(P) \otimes H^*(BG)$ whenever convenient. Similarly, the Universal coefficients theorem for cohomology, saying that $H^*(P) \otimes R \to H^*(P, R)$ is injective, allows us to work sometimes with $H^*(P) \otimes R$; see Remark 4.3. In this paper we are specially interested in the case of the cyclic group Z_q of prime order. This case is worked out in some detail in [7] and [13]. The reader can also trace back some of these ideas in the earlier papers by Jaworowski and Nakaoka. A somewhat different but related point of view is taken in the paper [2], where the emphasis is on the algebraic variety associated to the ideal Ind_G.

A useful index theory should be computable at least in the case of the most interesting G-spaces. The following proposition is an example of such a result. Many other computations can be found in the references.

PROPOSITION 5.4: *Suppose that $R^k \to E \to B$ is a real k-plane bundle and $S(E)$ is the associated sphere bundle. Assume that the group $G = Z_2$ acts on both E and $S(E)$ by the usual linear action and let $P(E) = S(E)/Z_2$ be the associated projective bundle. Let*

$$\mathrm{Ind}(S(E)) := \mathrm{Ker}\{H^*(B) \otimes H^*(BZ_2) \longrightarrow H^*(P(E))\},$$

where the cohomology is computed with Z_2-coefficients. Then the index $\mathrm{Ind}(S(E))$ *is a principal ideal in the ring* $H^*(B) \otimes H^*(BZ_2)$ *generated by the polynomial*

$$\Omega^k + w_1\Omega^{k-1} + w_2\Omega^{k-2} + \cdots + w_k,$$

where w_i *are the Stiefel–Whitney characteristic classes of the bundle* E *and* Ω *is the generator of the ring* $H^*(BZ_2; Z_2)$.

Proof: A Grothendieck approach to the theory of (Stiefel–Whitney) characteristic classes is based on the observation that the cohomology $H^*(P(E))$ of the projectivized bundle $P(E)$, associated to a vector bundle $R^k \to E \to B$, is a $H^*(B)$-algebra generated by a single generator t, subject to a single relation of the form $t^k + w_1t^{k-1} + \cdots + w_k$, [12]. The proposition follows directly from this observation. ∎

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